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THE STABILITY OF SPECTRAL SCHEMES FOR THE LARGE EDDY SIMULATION OF CHANNEL FLOWS

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SUMMARY

In a spectral LES code it is not possible to treat the actual eddy viscosity implicitly. We have therefore examined the effect on stability of adding a constant pseudo-viscosity to the implicit term and subtracting it from the explicit term: stability limits have been derived theoretically and verified computationally for two different treatments of the explicit term. We have also studied the effect of a stochastic temporal variation of the eddy viscosity.

KEY WORDS Numerical Stability Eddy Viscosity LES

INTRODUCTION

Numerical simulation is increasingly recognized as an attractive new method for studying turbulent flows. It is largely free from the uncertainty of closure models such as k–ε, while it can represent the flows in a detail that experiment cannot approach. Indeed some experimenters are now prepared to treat the output of numerical simulation codes as though it were experimental data.

The preferred method of simulation is to represent all sizes of structure (or eddy) down to the Kolmogorov length scales: the known properties of the energy spectrum at high wave number, including its exponential fall-off at high wave number, show that there is no need to go further. Since the simulation is necessarily three-dimensional and since the range of scales is large, the computation is extremely demanding even at the lowest Reynolds or Rayleigh numbers (*Re* or *Ra*) at which the flow is fully turbulent. Indeed, Corrsin[1] argued many years ago that such a 'full simulation' would never be possible.

Events have proved Corrsin wrong, but only just, and there is still no way of making a full simulation at *Re* or *Ra* values of practical interest. If we wish to make a numerical simulation of flows at even quite moderate *Re* or *Ra*, then we must turn to large eddy simulation (LES). In this technique the flow is divided into large eddies or grid scales (GS) and small eddies or subgrid scales (SGS): the former are represented, the latter are not. The unrepresented SGS interact with the represented GS through the non-linear terms in the equations of motion, the interaction appearing as an additional stress force $T_{ij,j}$. A full account is given by Voke and Collins.[2]

Since the SGS are not represented explicitly, the subgrid stress force must be modelled in terms of the GS only: this is known as subgrid modelling (SGM). The simplest SGM, still relevant today, is that of Smagorinsky,[3] in which

$$T_{ij}=\nu_e S_{ij}, \qquad S_{ij}=u_{i,j}+u_{j,i},$$
$$\nu_e=(C_{u1}\Delta)^2 S, \qquad S=(\tfrac{1}{2}S_{ij}S_{ji})^{1/2}. \tag{1}$$

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C_{u1} is a non-dimensional constant and Δ is the mesh spacing. Full details of this and later models are given by Voke and Collins.[2] With this model, the subgrid stress T_{ij} is not very different from the viscous stress νS_{ij}. However, the (molecular) kinematic viscosity ν is a property of the fluid (which is usually taken to be constant in LES), while the eddy viscosity ν_e must be computed and varies from place to place.

THE BUOYAN CODE

BUOYAN is one of the major codes of the Turbulence Unit (TU) of Queen Mary College. It solves the GS Navier–Stokes equations

$$\frac{\partial u_i}{\partial t}-\frac{\partial}{\partial x_j}\left((\nu+\nu_e)\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}-\frac{\partial}{\partial x_j}(u_i u_j), \tag{2}$$

u_i being the GS velocity. (The meaning of p is not straightforward and the SGM is more elaborate than this equation suggests; these points are not relevant to the present discussion.) An account of the code is given by Gavrilakis *et al.*[4]

The code is limited to fully developed flow in a parallel-sided channel, with periodicity in the streamwise and spanwise directions. It is fully spectral, using a Fourier representation in these two directions and Chebyshev polynomials normal to the wall. The molecular-viscous and pressure terms are treated implicitly (Crank–Nicholson), while the eddy-viscous and inertial terms are treated explicitly (Adams–Bashforth). The whole time advancement is thus second-order accurate, and the code can override the molecular-viscous limit on the time step. It would be possible to override the advective Courant limit, but we believe that important information would be lost if this were done.

Thus the time advancement equations are implemented as

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\tfrac{1}{2}(M_i^{n+1}+M_i^n)+(\tfrac{3}{2}H_i^n-\tfrac{1}{2}H_i^{n-1}), \tag{3}$$

where

$$M_i=(\text{transform of})\left(\frac{\partial}{\partial x_j}(\nu S_{ij}-p\delta_{ij})\right), \tag{4}$$

$$H_i=(\text{transform of})\left(\frac{\partial}{\partial x_j}(\nu_e S_{ij}-u_i u_j)\right), \tag{5}$$

while the affix indicates the time level. To solve (3), BUOYAN uses a fast projective-type algorithm due to Antonopoulos–Domis, in which there is no need to solve a Poisson equation for the pressure.

THE EDDY VISCOSITY LIMIT

The matrix representation of the molecular viscosity is diagonal in Fourier space and tridiagonal in Chebyshev space: thus the implicit treatment of this effect is compatible with a fast solution for u_i^{n+1}, the velocity field at the new time level. None of this is true for the eddy viscosity. Since it is space-dependent, its matrix representation is full: the process of solving for u_i^{n+1} would have been slowed down very greatly if this term had been treated implicitly. The consequence of (5) is that the code cannot override the eddy-viscous limit, and this could result in a considerable wastage of

computer time. It should be emphasized that this problem is unique to spectral codes: the eddy viscosity is diagonal in finite difference codes.

We have therefore added, as an option in the BUOYAN code, the facility to add a constant eddy viscosity to the implicit term and to subtract it from the explicit term. The full forms of equations (4) and (5) are thus

$$M_i = (\text{transform of}) \left(\frac{\partial}{\partial x_j} [(\nu + \bar{\nu}_e) S_{ij} - p\delta_{ij}] \right), \tag{6}$$

$$H_i = (\text{transform of}) \left(\frac{\partial}{\partial x_j} [(\nu_e - \bar{\nu}_e) S_{ij} - u_i u_j] \right), \tag{7}$$

$\bar{\nu}_e$ being at the code user's discretion. Since it is constant, fast solution is still possible.

One may hope that, with a suitably chosen value of $\bar{\nu}_e$, this option will enable the user to override the eddy-viscous limit. The theme of this paper is the stability analysis of the Crank–Nicholson/Adams–Bashforth (CN–AB) scheme set out in equation (3), to test whether the option of including a constant eddy viscosity does achieve its intended purpose. Since some workers (e.g. Schumann[5]) have preferred LeapFrog to Adams–Bashforth, we have also analysed the Crank–Nicholson/LeapFrog (CN–LF) scheme

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \tfrac{1}{2}(M_i^{n+1} + M_i^{n-1}) + H_i^n. \tag{8}$$

(This formulation of the CN term assures stability in the absence of an explicit term.)

As has been said above, there would be no reason to introduce the modification (6) and (7) into a finite difference code. We have nonetheless done the analysis in a (one-dimensional) finite difference framework, because this simplified the work very considerably. From past experience, one can be confident that the results so deduced will also apply to spectral codes: the quantity Δx is to be interpreted as the spacing between the collocation points of the spectral scheme.

STABILITY ANALYSIS

CN–AB scheme

From equations (3), (6) and (7) the diffusion equation for one dimension, say x, may be written

$$\frac{u^{n+1} - u^n}{\Delta t} = (\nu + \bar{\nu}_e)(\tfrac{1}{2} u_{xx}^{n+1} + \tfrac{1}{2} u_{xx}^n) + (\nu - \bar{\nu}_e)(\tfrac{3}{2} u_{xx}^n - \tfrac{1}{2} u_{xx}^{n-1}). \tag{9}$$

In the von Neumann formalism (see e.g. Roache[6])

$$u_{xx}^n = -2(1 - \cos\theta)\frac{u^n}{\Delta x^2}, \qquad \theta = k\Delta x, \tag{10}$$

and (9) reduces to

$$u^{n+1} = bu^n + cu^{n-1}, \tag{11}$$

with

$$b=\frac{1-\frac{1}{2}e_1-\frac{3}{2}e_2}{1+\frac{1}{2}e_2}, \qquad c=\frac{\frac{1}{2}e_2}{1+\frac{1}{2}e_1}, \tag{12}$$

$$e_r=2(1-\cos\theta)d_r, \qquad r=1,2, \tag{13}$$

$$d_1=\frac{(v+\bar{v}_e)\Delta t}{\Delta x^2}, \qquad d_2=\frac{(v_e-\bar{v}_e)\Delta t}{\Delta x^2}. \tag{14}$$

To analyse (11), put

$$u^{n+1}=\lambda u^n$$

and then

$$\lambda^2-b\lambda-c=0. \tag{15}$$

The stability requirement is simply

$$|\lambda|<1. \tag{16}$$

From the results proved by Miller[7] (we are indebted to a referee for this reference) it can be easily shown that the necessary and sufficient stability conditions for equation (15) are

$$|c|<1 \tag{17}$$

and

$$|b|+c<1. \tag{18}$$

Equation (18) implies

$$e_1+e_2>0 \tag{19}$$

and

$$e_2<1, \tag{20}$$

and equation (17) implies

$$|e_2|<2+e_1, \tag{21}$$

which must hold if (19) and (20) are satisfied. Equation (19) is automatically satisfied, so that equation (20) is all that is needed.

Since

$$\left.\begin{matrix}\max\\ \min\end{matrix}\right\} 2(1-\cos\theta)=\begin{cases}4\\ 0\end{cases}, \tag{22}$$

we must require (to be sure of complying with equation (20))

$$d_2<\tfrac{1}{4} \tag{23}$$

or

$$\frac{v_e-\bar{v}_e}{\Delta x^2}\Delta t<\frac{1}{4}, \tag{24}$$

which shows why it might be advantageous to use a non-zero $\bar{v}_e$. If we let $\bar{v}_e=0$ (so that no use is

made of the volume-averaging facility in BUOYAN), then (24) reduces to

$$\frac{\nu_e \Delta t}{\Delta x^2} < \frac{1}{4}, \tag{25}$$

which is independent of the value of e_1 and is therefore the known viscous stability limit for a pure Adams–Bashforth scheme. It is twice as tight as that for a simple Euler scheme, and this is presumably because the Adams–Bashforth scheme spans a double time interval. Obviously the requirement for (24) is weaker than (25). If we choose $\bar{\nu}_e > \nu_e$, (24) is always satisfied, so the computation will always be stable.

From the above analysis the conclusions on the CN–AB scheme are that if we do not use the $\bar{\nu}_e$ option, the diffusive limit (25) on the Adams–Bashforth scheme must be observed; it is quite restrictive. If we use the option and let $\bar{\nu}_e < \nu_e$, (24) must be satisfied in order to make sure of the stability. However, if we choose $\bar{\nu}_e > \nu_e$, the calculation will be always stable.

The above conclusions can easily be extended to three-dimensional calculations. For example, equation (24) generalizes in the obvious way to

$$(\nu_e - \bar{\nu}_e)\,\Delta t\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}\right) < \frac{1}{4}. \tag{26}$$

CN–LF scheme

Equation (8) can be simplified to

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = (\nu + \bar{\nu}_e)\left(\tfrac{1}{2} u_{xx}^{n+1} + \tfrac{1}{2} u_{xx}^{n-1}\right) + (\nu - \bar{\nu}_e)\, u_{xx}^{n} \tag{27}$$

for one-dimensional diffusion. This can be put into the form (11)

$$u^{n+1} = bu^n + cu^{n-1} \tag{11}$$

and now

$$b = -\frac{2e_2}{1+e_1}, \qquad c = \frac{1-e_1}{1+e_1}, \tag{28}$$

e_1 and e_2 being defined in (13) and (14).

The stability criteria are thus

$$|b| + c < 1 \quad \text{if } b^2 + 4c > 0, \tag{29}$$

$$|c| < 1 \quad \text{if } b^2 + 4c < 0. \tag{30}$$

(29) reduces to

$$e_1 > |e_2| \tag{31}$$

Since $e_1 = 0$, $e_2 > 0$ in the pure LeapFrog scheme, the condition (31) cannot be satisfied. Thus the pure LeapFrog treatment of pure diffusion is absolutely unstable, and this is another known result. For the mixed CN–LF scheme the effect of these relations is as follows.

When $b^2 + 4c > 0$, we find:

(i) If $\nu > \nu_e$, the calculation is stable for any Δt with $\bar{\nu}_e = 0$.
(ii) If $\bar{\nu}_e > \nu_e$ and ν, then the calculation is automatically stable.
(iii) If $\nu_e > \nu$ and $\bar{\nu}_e$, then it is stable for

$$\bar{\nu}_e > \tfrac{1}{2}(\nu_e - \nu). \tag{32}$$

When $b^2+4c<0$, then, according to (28), the requirement (30) is always satisfied. Then we have automatic stability.

To summarize, if $v>v_e$, the calculation is stable for any Δt with $\bar{v}_e=0$; if we cannot be sure of satisfying $v>v_e$, then $\bar{v}_e$ should be chosen to satisfy (32).

Thus the behaviour of the CN–LF scheme is distinctly better than that of the CN–AB scheme, since the latter never allows an unlimited time step with $\bar{v}_e=0$. Nonetheless, the CN–AB scheme can be stabilized by choosing $\bar{v}_e>v_e$.

NUMERICAL TESTS OF THE STABILITY ANALYSIS

The conclusions drawn from the stability analysis above have been tested by computation. The results are wholly consistent with the analytical conclusions.

As a typical example, we have investigated the one-dimensional unsteady flow between two flat plates: this is illustrated schematically in Figure 1. The upper plate is impulsively given a constant speed while the lower plate is kept stationary. The aim is to find the velocity distribution between the plates as a function of time.

The form of equation (9) appropriate to this problem is

$$c_{j1}u_{j-1}^{n+1}+c_{j2}u_j^{n+1}+c_{j3}u_{j+1}^{n+1}=c_{j4}, \tag{33}$$

with

$$\begin{aligned} &c_{j1}=d_1, \qquad c_{j2}=-(2+2d_2), \qquad c_{j3}=d_1,\\ &c_{j4}=-(d_1+3d_2)\,u_{j-1}^n-(2-2d_1-6d_2)\,u_j^n-(d_1+3d_2)\,u_{j+1}^n\\ &\quad +d_2(u_{j-1}^{n-1}-2u_j^{n-1}+u_{j+1}^{n-1}), \end{aligned} \tag{34}$$

d_1 and d_2 being defined by (14).

Solving this tridiagonal matrix problem, we can get the velocity distribution between the two plates at different times. Some typical results are shown in Figures 2 and 3.

From (27) the CN–LF scheme for this problem can be formulated as

$$c_{j1}u_{j-1}^{n+1}+c_{j2}u_j^{n+1}+c_{j3}u_{j+1}^{n+1}=c_{j4}, \tag{35}$$

where

$$\begin{aligned} &c_{j1}=d_1, \qquad c_{j2}=-(1+2d_1), \qquad c_{j3}=d_1,\\ &c_{j4}=-2d_2(u_{j-1}^n-2u_j^n+u_{j+1}^n)-d_1u_{j-1}^{n-1}-(1-2d_1)u_j^{n-1}+d_1u_{j+1}^{n-1}. \end{aligned} \tag{36}$$

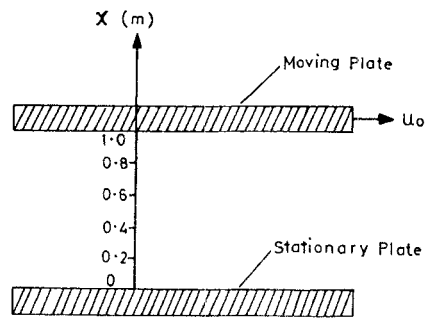

Figure 1. Schematic diagram of moving plate

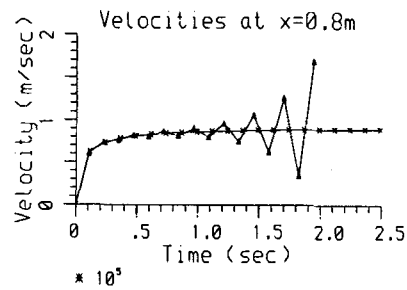

Figure 2. Verification of stability criterion of equation (24) for CN–AB scheme:
—×—, $d_2 = 0{\cdot}20 < \frac{1}{4}$, stable;
—Δ—, $d_2 = 0{\cdot}28 > \frac{1}{4}$, unstable

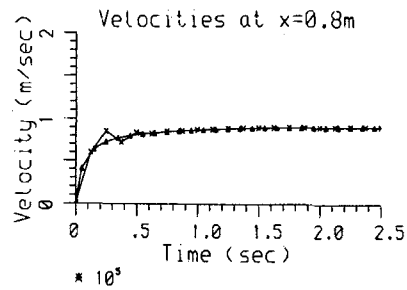

Figure 3. Verification of stability criterion for CN–AB scheme with $\bar{v}_e > v_e$:
—×—, $d_2 = -1{\cdot}0$, stable;
—Δ—, $d_2 = -0{\cdot}2$, stable

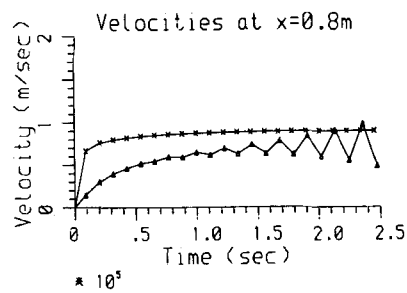

Figure 4. Verification of stability for CN–LF scheme with $\bar{v}_e = 0$:
—×—, $d_2 = 0{\cdot}6$ when $v > v_e$, stable;
—Δ—, $d_2 = 0{\cdot}1$ when $v < v_e$, unstable

The different sets of v, v_e, $\bar{v}_e$ and d_2 are again chosen so as to check the stability of the scheme: typical results are shown in Figures 4 and 5.

We have performed calculations for all the stability criteria found analytically for two finite difference schemes, but for simplicity only some typical results are presented here. The computation confirms the analysis in every case.

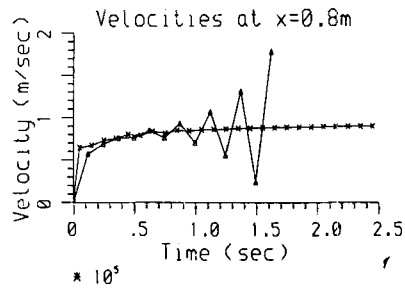

Figure 5. Verification of stability criterion of equation (32) for CN–LF scheme with $\bar{v}_e > 0$:
—×—, $d_2 = 0{\cdot}4$ when $\bar{v}_e > \frac{1}{2}(v_e - \nu)$, stable;
—△—, $d_2 = 0{\cdot}1$ when $\bar{v}_e < \frac{1}{2}(v_e - \nu)$, unstable

EFFECT OF TIME-VARYING EDDY VISCOSITY

The eddy viscosity is formed from stochastic variables and therefore the stability parameter λ will vary from one time step to the next. When λ is constant, the solution of the pure decay problem is

$$u^n = \lambda^n u^0. \tag{37}$$

Provided $|\lambda| < 1$, the solution will converge to zero as it should. This is clearly necessary for the proper simulation of the driven problem with a stochastic input, but it is not so clear that it is sufficient.

In the random problem (37) generalizes to

$$u^n = \left(\prod_{j=1}^{n} \lambda_j \right) u^0 \tag{38}$$

and we now require that the product should tend to zero as n tends to infinity. It can be proved that a sufficient condition for this is

$$\frac{1}{n} \sum_{j=1}^{n} \lambda < 1 \quad \text{for large } n. \tag{39}$$

Since (39) is sufficient, then the implication is that the stability criteria may be applied to the mean value of v_e (that is, the local time mean). Upward fluctuations which badly violate the stability limit may temporarily disturb the simulation, but it should recover quite quickly. This is proved by computational tests in which the input value of v_e was generated from an exponential distribution with a chosen v_e (mean).

Now we define

$$d_2(\text{mean}) = \frac{v_e(\text{mean}) - \bar{v}_e}{\Delta x^2} \Delta t. \tag{40}$$

For the CN–AB scheme when $\bar{v}_e = 0$, the stability condition, according to equation (25), should now be

$$\frac{v_e(\text{mean})}{\Delta x^2} \Delta t < \frac{1}{4}, \tag{41}$$

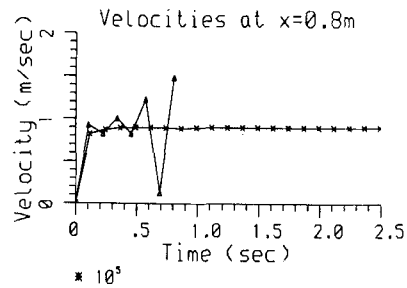

Figure 6. Stability tests with random ν_e for CN–AB scheme:
—×—, d_2(mean)=0·25 with $\bar{\nu}_e=0$, stable;
—△—, d_2(mean)=0·34 with $\bar{\nu}_e=0$, unstable

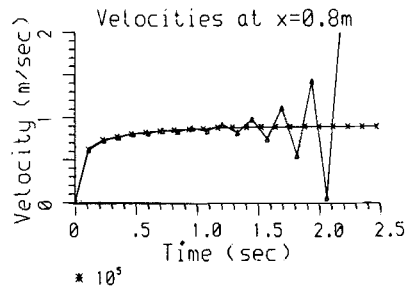

Figure 7. The comparison of stabilities with ν_e(mean) and ν_e for CN–AB scheme:
—×—, d_2(mean)=0·27 with $\bar{\nu}_e=0$, stable;
—△—, d_2=0·27 with $\bar{\nu}_e=0$, unstable

and some test results are given in Figure 6. We have also performed numerous other tests for different schemes and they always confirm the conclusions drawn above.

It is interesting that in some cases with a non-zero ν_e(mean) the stochastic computation may be stable, while one with a constant ν_e equal to the mean value of the stochastic computation may be unstable. Such behaviour is shown in Figure 7.

Of course one cannot ignore the variation of the local time mean of ν_e with z, and the criterion (26) must be applied at that z-value where its left-hand side is largest.

CONCLUSIONS

The numerical stabilities of the CN–AB and CN–LF schemes have been analysed and complete stability criteria have been obtained. The results show that inclusion of the option of a constant eddy viscosity in the BUOYAN code eases the stability requirement: if $\bar{\nu}_e$ is carefully chosen the calculations with either scheme can be made stable with any time step. These conclusions have been carefully verified by computation tests. The effect of a time-varying eddy viscosity on stability is investigated and the sufficient condition for stability with random eddy viscosity inputs has been theoretically obtained and computationally verified. Although the analysis has been done for the one-dimensional situation only, the conclusions can be easily extended to three dimensions.

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